On the Bézoutian for Polynomial Matrices

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Dedicated to Helmut Wielandt on his 75th birthday

Submitted by Thomas J. Laffey

ABSTRACT

The Bézoutian B of two polynomial matrices can be described as a solution of a linear matrix equation. This fact yields a new proof of the Barnett factorization of B.

Let $F(z)=z^sI-(A_{s-1}z^{s-1}+\cdots+A_1z+A_0)\in\mathbb{C}^{k\times k}[z]$ and $G(z)=\sum_{r=0}^{s}G_rz^r\in\mathbb{C}^{m\times m}[z]$ be two polynomial matrices. The Bézoutian $B=\operatorname{Bez}(F,G)$ of F and G (see [1]) is the skm-size matrix with partitions B_{ij} defined by

$$\sum_{i,j=0}^{s-1}B_{ij}x^iy^j=\frac{1}{x-y}\left[F(x)\otimes G(y)-F(y)\otimes G(x)\right].$$

The first author has shown in [4] that the Bézoutian of two scalar polynomials satisfies a linear matrix equation. We will first extend this result to polynomial matrices.

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Our notation will be the following. We associate with F the block companion matrices C_F and C:

$$C := C_F \otimes I_m, \qquad C_F := \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{s-1} \end{pmatrix}.$$

If N is given by

$$N := \begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{pmatrix}_{s \times s},$$

we put

$$S := N \otimes I_{km}.$$

We define

$$p(x) = (1, x, x^2, \dots, x^{s-1})^T \quad \text{and} \quad P(x) = p(x) \otimes I_{km}.$$

With this notation we have

$$P^T(x) B P(y) = \sum_{i,j=0}^{s-1} B_{ij} x^i y^j.$$

Let $e_0^T = (1,0,\dots,0), \dots,$ $e_{s-1}^T = (0,\dots,0,1)$ be the unit s-vectors and

$$E_i^T := e_i^T \otimes I_k.$$

For $d(x,y) = \sum_{i,j=0}^{s-1} d_{ij} x^i y^j \in \mathbb{C}^{a \times b}[x,y]$ and $r \in \mathbb{N}$ we put

$$\prod_x \frac{1}{x^r} d = \sum_{i \geq r, j=0}^{s-1} d_{ij} x^{i-r} y^j.$$

Note that

$$e_{s-1} = \prod_x \frac{1}{x^{s-1}} p(x). \tag{1}$$

LEMMA 1. *The matrix* $B = \mathrm{Bez}(F, G)$ *satisfies the equation*

$$B - SBC = \begin{pmatrix} -A_1 \otimes I_m \\ \vdots \\ -A_{s-1} \otimes I_m \\ I_{km} \end{pmatrix} (\tilde{G}_0, \dots, \tilde{G}_{s-1}), \tag{2}$$

where $\tilde{G}_i$ *is given by*

$$\tilde{G}_i := I_k \otimes G_i + A_i \otimes G_s. \tag{3}$$

Proof. We focus on

$$M(x, y) := P^T(y)(B - SBC)P(x).$$

It is not difficult to verify that $P^T(y)S = y^{-1}[P^T(y) - P^T(0)]$ and $CP(x) = xP(x) - e_{s-1} \otimes F(x) \otimes I_m$. Hence

$$M(x, y) = y^{-1}\{ -(x - y)P^T(y)BP(x) + xP^T(0)BP(x) + [P^T(y) - P^T(0)] B [e_{s-1} \otimes F(x) \otimes I_m] \}.$$

Obviously $P^T(0)BP(x) = x^{-1}[F(x) \otimes G(0) - F(0) \otimes G(x)]$. Taking (1) into account, we get

$$\begin{aligned} P^T(y)B(e_{s-1} \otimes I_{km}) &= \prod_x x^{-s+1} P^T(y)BP(x) \\ &= \prod_x x^{-s}\left(1 - \frac{y}{x}\right)^{-1} [F(x) \otimes G(y) - F(y) \otimes G(x)] \\ &= I \otimes G(y) - F(y) \otimes G_s \end{aligned}$$

and

$$
\begin{aligned}
&[P^T(y) - P^T(0)]\, B\, [e_{s-1} \otimes F(x) \otimes I_m] \\
&\qquad = \{ I \otimes [G(y) - G(0)] - [F(y) - F(0)] \otimes G_s \} [F(x) \otimes I_m] \\
&\qquad = F(x) \otimes [G(y) - G(0)] - [F(y) - F(0)]\, F(x) \otimes G_s .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
M(x,y) &= y^{-1} \{ [F(y) - F(0)] \otimes I_m \} \{ I_k \otimes G(x) - F(x) \otimes G_s \} \\
&= P^T(y) \begin{pmatrix} -A_1 \otimes I_m \\ \vdots \\ -A_{s-1} \otimes I_m \\ I_{km} \end{pmatrix} (\tilde{G}_0, \dots, \tilde{G}_{s-1}) P(x)
\end{aligned}
$$

with $\tilde{G}_i$ given by (3). ■

The preceding lemma leads to the Barnett factorization of B. We define

$$
G(C_F) := \sum_{i=0}^{s} C_F^i \otimes G_i .
$$

THEOREM [1]. *The Bézoutian* $B = \mathrm{Bez}(F, G)$ *can be written as*

$$
B = T G(C_F),
$$

where $T := T_F \otimes I_m$ *and*

$$
T_F := \begin{pmatrix} -A_1 & -A_2 & \cdot & \cdot & \cdot & -A_{s-1} & I \\ -A_2 & & & & \cdot & I & \\ \cdot & & & \cdot & \cdot & & \\ \cdot & & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & & \\ -A_{s-1} & \cdot & & & & & \\ I & & & & & & \end{pmatrix} .
$$

Proof. As a solution of (2) with $S^s = 0$ the matrix B is given by the finite series

$$B = \sum_{i=0}^{s-1} S^i \begin{pmatrix} -A_1 \otimes I \\ \vdots \\ -A_{s-1} \otimes I \\ I_{km} \end{pmatrix} (\tilde{G}_0, \dots, \tilde{G}_{s-1}) C^i$$

$$= T \begin{pmatrix} W \\ WC \\ \vdots \\ WC^{s-1} \end{pmatrix},$$

where

$$W := (\tilde{G}_0, \dots, \tilde{G}_{s-1}) = \sum_{j=0}^{s-1} e_j^T \otimes \tilde{G}_j$$

$$= \sum_{j=0}^{s-1} E_j^T \otimes G_j + \sum_{j=0}^{s-1} A_j E_j^T \otimes G_s.$$

The matrix C_F satisfies $E_j^T = E_0^T C_F^j$, $j = 0, \dots, s-1$, and

$$\begin{pmatrix} E_0^T \\ \vdots \\ E_0^T C_F^{s-1} \end{pmatrix} = I_{skm}. \tag{4}$$

Therefore

$$\left(E_j^T \otimes G_j + A_j E_j^T \otimes G_s \right) C^i = E_0^T C_F^{j+i} \otimes G_j + A_j E_0^T C_F^{j+i} \otimes G_s$$

$$= \left(E_0^T C_F^i \otimes I_m \right) \left(C_F^j \otimes G_j \right) + \left[A_j \left(E_0^T C_F^i \right) C_F^j \right] \otimes G_s,$$

and (4) yields

$$\begin{pmatrix} W \\ \vdots \\ WC^{s-1} \end{pmatrix} = \sum_{j=0}^{s-1} \left(C_F^j \otimes G_j \right) + \sum_{j=0}^{s-1} \left(I_s \otimes A_j \right) C_F^j \otimes G_s. \tag{5}$$

Form a block Cayley-Hamilton theorem,

$$C_F^s - \sum_{j=0}^{s-1} (I_s \otimes A_j) C_F^j = 0,$$

we obtain $\sum_{j=0}^{s} C_F^j \otimes G_j$ for (5). ■

In [2] a generalized Bézoutian is defined for strictly proper rational matrices. In the case where F and G are of the same size, that generalized Bézoutian does not coincide with $\mathrm{Bez}(F, G)$. A different proof of the Barnett factorization of $B(F, G)$ can be found in [3].

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